

Some applications of τ -tilting theory^{*}

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Abstract

Let A be a finite dimensional algebra over an algebraically closed field k , and M be a partial tilting A -module. We prove that the Bongartz τ -tilting complement of M coincides with its Bongartz complement, and then we give a new proof of that every almost complete tilting A -module has at most two complements. Let $A = kQ$ be a path algebra. We prove that the support τ -tilting quiver $\vec{Q}(\text{stilt}A)$ of A is connected. As an application, we investigate the conjecture of Happel and Unger in [9] which claims that each connected component of the tilting quiver $\vec{Q}(\text{tilt}A)$ contains only finitely many non-saturated vertices. We prove that this conjecture is true for Q being all Dynkin and Euclidean quivers and wild quivers with two or three vertices, and we also give an example to indicate that this conjecture is not true if Q is a wild quiver with four vertices.

Key words and phrases: τ -tilting module, support τ -tilting quiver, tilting module, tilting quiver.

1 Introduction

Adachi, Iyama and Reiten introduce τ -tilting theory which completes the classical tilting theory from the viewpoint of mutation in [1], and they establish a bijection between the tilting objects in a cluster category and the support τ -tilting modules over each cluster-tilted algebra.

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As a generalization of classical tilting modules, support τ -tilting modules satisfy many nice properties. For example, every basic almost complete support τ -tilting module is the direct summand of exactly two basic support τ -tilting modules. This means that mutation of support τ -tilting modules is always possible. Moreover, the set of support τ -tilting modules has a natural structure of poset and the Hasse quiver of this poset coincides with the mutation quiver of support τ -tilting modules. It is also known that there are close relations between support τ -tilting modules, functorially finite torsion classes and two-term silting complexes, see [1] for details.

In this paper, we use the properties of support τ -tilting modules to prove that the Bongartz τ -tilting complement of a partial tilting module coincides with its Bongartz complement, and then we give a new proof of that every almost complete tilting A -module has at most two complements. As an application, we prove that the support τ -tilting quiver $\vec{Q}(\text{stilt}A)$ of A is connected if A is hereditary. Moreover, we investigate the conjecture of Happel and Unger in [9] which claims that each connected component of the tilting quiver $\vec{Q}(\text{tilt}A)$ contains only finitely many non-saturated vertices. We prove that this conjecture is true for Q being all Dynkin and Euclidean quivers and wild quivers with two or three vertices, and we also give an example to indicate that this conjecture is not true if Q is a wild quiver with four vertices.

Let A be a finite dimensional algebra over an algebraically closed field k . For an A -module M , we denote by $|M|$ the number of pairwise nonisomorphic indecomposable direct summands of M .

An A -module T is called a tilting module if it satisfies the following conditions:

- (1) $\text{pd}_A T \leq 1$;
- (2) $\text{Ext}_A^1(T, T) = 0$;
- (3) There is a short exact sequence $0 \rightarrow A \rightarrow T_1 \rightarrow T_2 \rightarrow 0$ with $T_1, T_2 \in \text{add } T$.

An A -module M satisfying the above conditions (1) and (2) is called a partial tilting module and if moreover $|M| = |A| - 1$, then M is called an almost complete tilting module.

The following definition is taken from [1].

Definition 1. (a) An A -module M is called τ -rigid if $\text{Hom}_A(M, \tau M) = 0$.

(b) An A -module M is called τ -tilting (respectively almost complete τ -tilting) if M is

τ -rigid and $|M| = |A|$ (respectively $|M| = |A| - 1$).

(c) An A -module M is called support τ -tilting if there exists an idempotent e in A such that M is a τ -tilting $(A/\langle e \rangle)$ -module.

From the above definition we know that any tilting (partial tilting) A -module M is τ -tilting (τ -rigid). Let M be a partial tilting A -module. By [1, Theorem 2.10] there exists a τ -rigid A -modules X such that $M \oplus X$ is a τ -tilting A -module and $\text{Fac}(M \oplus X) = {}^\perp(\tau M)$. X is called the Bongartz τ -tilting complement of M . The partial tilting A -module M also has a Bongartz complement. We prove the following theorem.

Theorem A. *Let M be a partial tilting A -module and X be its Bongartz τ -tilting complement. Then $\text{pd}_A X \leq 1$ and $T = M \oplus X$ is a tilting A -module. In particular, X coincides with the Bongartz complement of M .*

D.Happel and L.Unger prove in [6] that for an almost complete tilting A -module M , it has exactly two nonisomorphic complements if and only if M is faithful. In this paper, we give a new proof of this theorem from the viewpoint of mutation of support τ -tilting modules.

Tiling quiver $\vec{Q}(\text{tilt} A)$ is introduced in [15] by Riedtmann and Schofield, which gives an explicit description of relations between tilting modules. Also Adachi, Iyama and Reiten define the support τ -tilting quiver $\vec{Q}(s\tau\text{-tilt} A)$ in [1]. We prove that the tilting quiver $\vec{Q}(\text{tilt} A)$ can be embedded into the support τ -tilting quiver $\vec{Q}(s\tau\text{-tilt} A)$. Then we calculate the number of arrows in $\vec{Q}(\text{tilt} A)$ when $A = kQ$ is a Dynkin hereditary algebra and show that the number of arrows in $\vec{Q}(\text{tilt} A)$ is independent of the orientation of Q . It is known that $\vec{Q}(\text{tilt} A)$ may not be connected when A is a hereditary algebra. But for $\vec{Q}(s\tau\text{-tilt} A)$, we give the following result.

Theorem B. *Let A be a finite dimensional hereditary algebra. Then the support τ -tilting quiver $\vec{Q}(s\tau\text{-tilt} A)$ is connected.*

Assume $A = kQ$ is a finite dimensional hereditary algebra. Note that the tilting quiver $\vec{Q}(\text{tilt} A)$ may contain several connected components. A conjecture of Happel and Unger in [9] is that each connected component of $\vec{Q}(\text{tilt} A)$ contains finitely many non-saturated vertices. We prove that this conjecture is true for Q being all Dynkin and

Euclidean quivers and wild quivers with two or three vertices.

Theorem C. *Let $A = kQ$ be a finite dimensional hereditary algebra. If Q is a Dynkin quiver, a Euclidean quiver or a wild quiver with two or three vertices, then each connected component of the tilting quiver $\vec{Q}(\text{tilt } A)$ contains finitely many non-saturated vertices.*

Remark. *Let $Q : 1 \rightleftharpoons 2 \leftarrow 3 \rightarrow 4$ and $B = kQ$. We will show that the tilting quiver $\vec{Q}(\text{tilt } B)$ contains a connected component which has infinitely many non-saturated vertices. Therefore, the conjecture of Happel and Unger is not true for some wild quivers.*

This paper is arranged as follows. In section 2, we fix the notations and recall some necessary facts needed for our research. In section 3, we prove Theorem A. Section 4 and section 5 are devoted to the proof of Theorem B and Theorem C respectively.

2 Preliminaries

Let A be a finite dimensional algebra over an algebraically closed field k . We denote by $\text{mod-}A$ the category of all finitely generated right A -modules and by $D = \text{Hom}_k(-, k)$ the standard duality between $\text{mod-}A$ and $\text{mod-}A^{op}$. We denote by τ_A the Auslander-Reiten translation of A .

Given any A -module M , $\text{Fac } M$ is the subcategory of $\text{mod-}A$ whose objects are generated by M and $\text{add } M$ is the subcategory of $\text{mod-}A$ whose objects are the direct summands of finite direct sums of copies of M . We denote by M^\perp (respectively ${}^\perp M$) the subcategory of $\text{mod-}A$ with objects $X \in \text{mod-}A$ satisfying $\text{Hom}_A(M, X) = 0$ (respectively $\text{Hom}_A(X, M) = 0$). $\text{pd}_A M$ is the projective dimension of M . We decompose M as $M \cong \bigoplus_{i=1}^m M_i^{d_i}$, where each M_i is indecomposable, $d_i > 0$ for any i and M_i is not isomorphic to M_j if $i \neq j$. The module M is called *basic* if $d_i = 1$ for any i . If M is basic, we define $M[i] = \bigoplus_{j \neq i} M_j$.

For τ -tilting modules, we have the following result in [1].

Lemma 2.1. [1, Proposition 1.4] *Any faithful τ -tilting A -module is a tilting A -module.*

Some certain pairs of A -modules are introduced in [1], and it is convenient to view τ -rigid modules and support τ -tilting modules as these pairs.

Definition 2.1. Let (M, P) be a pair with $M \in \text{mod-}A$ and $P \in \text{proj-}A$.

- (a) We call (M, P) a τ -rigid pair if M is τ -rigid and $\text{Hom}_A(P, M) = 0$.
- (b) We call (M, P) a support τ -tilting (respectively almost complete support τ -tilting) pair if (M, P) is a τ -rigid pair and $|M| + |P| = |A|$ (respectively $|M| + |P| = |A| - 1$).

(M, P) is called basic if M and P are basic and we say (M, P) is a direct summand of (M', P') if M is a direct summand of M' and P is a direct summand of P' . One of the main results in [1] is the following.

Lemma 2.2. [1, Theorem 2.18] Any basic almost complete support τ -tilting pair (U, Q) is a direct summand of exactly two basic support τ -tilting pairs (T, P) and (T', P') .

Then (T, P) is called left mutation of (T', P') if $\text{Fac } T \subseteq \text{Fac } T'$ and this is denoted by $T = \mu^-(T')$. Adachi, Iyama and Reiten show in [1] that one can calculate left mutations of support τ -tilting modules by exchange sequence constructed from left approximations.

Lemma 2.3. [1, Theorem 2.30] Let $T = X \oplus U$ be a basic τ -tilting A -module where the indecomposable A -module X is the Bongartz τ -tilting complement of U . Let $X \xrightarrow{f} U' \xrightarrow{g} Y \rightarrow 0$ be an exact sequence where f is a minimal left U -approximation. Then we have the following.

- (a) If U is not sincere, then $Y=0$. In this case $U=\mu_X^-(T)$ holds and it is a basic support τ -tilting A -module which is not τ -tilting.
- (b) If U is sincere, then Y is a direct sum of copies of an indecomposable A -module Y_1 and $Y_1 \notin \text{add } T$. In this case $Y_1 \oplus U = \mu_X^-(T)$ holds and it is a basic τ -tilting A -module.

The support τ -tilting quiver $\vec{Q}(\text{s}\tau\text{-tilt } A)$ is defined as follows:

Definition 2.2. (1) The set of vertices is $\text{s}\tau\text{-tilt } A$

- (2) There is an arrow from T to U if U is a left mutation of T .

Since we have a bijection $T \rightarrow \text{Fac } T$ between basic support τ -tilting modules and functorially finite torsion classes, there exists a natural partial order on the set $\text{s}\tau\text{-tilt } A$ of support τ -tilting A -modules: $T_1 < T_2$, if $\text{Fac } T_1 \subseteq \text{Fac } T_2$. Moreover, the Hasse quiver of this poset coincides with the support τ -tilting quiver $\vec{Q}(\text{s}\tau\text{-tilt } A)$.

The following lemma in [1] is very useful.

Lemma 2.4. [1, Lemma 2.20] *Let (T, P) be a τ -rigid pair for A and $P(\text{Fac}T)$ be the direct sum of one copy of each indecomposable Ext-projective A -modules in $\text{Fac}T$. If U is a τ -rigid A -module satisfying ${}^\perp(\tau T) \cap P^\perp \subseteq {}^\perp(\tau U)$, then there is an exact sequence $U \xrightarrow{f} T' \rightarrow C \rightarrow 0$ satisfying the following conditions*

- (1) *f is a minimal left $\text{Fac}T$ -approximation.*
- (2) *$T' \in \text{add}T$, $C \in \text{add}P(\text{Fac}T)$ and $\text{add}T' \cap \text{add}C = 0$.*

Let A be a finite dimensional hereditary algebra and \mathcal{C}_A be the cluster category associated to A . We assume that \mathcal{C}_A has a cluster-tilting object T and $\Lambda = \text{End}_{\mathcal{C}}(T)$ is the cluster-tilted algebra. We have the following.

Lemma 2.5. [1, Theorem 4.1] *There exists a bijection between basic cluster tilting objects in \mathcal{C}_A and the basic support τ -tilting modules over Λ*

Assume $A = kQ$ is a finite dimensional hereditary algebra where Q is a finite quiver with n vertices and $a_s(Q)$ ($1 \leq s \leq n$) denote the number of basic support τ -tilting A -modules with s nonisomorphic indecomposable direct summands. Note that the support τ -tilting A -modules coincide with the support tilting A -modules since A is a hereditary algebra. If Q is a Dynkin quiver, according to [13], all $a_s(Q)$ ($1 \leq s \leq n$) are constants and do not depend on the orientation of Q .

Lemma 2.6. [13, Theorem 1] *Let $A = kQ$ be a path algebra of a Dynkin quiver Q . Then we have*

Q	A_n	D_n	E_6	E_7	E_8
$a_n(Q)$	$\frac{1}{n+1}C_{2n}^n$	$\frac{3n-4}{2n-2}C_{2n-2}^{n-2}$	418	2431	17342
$a_{n-1}(Q)$	$\frac{2}{n+1}C_{2n-1}^{n-1}$	$\frac{3n-4}{2n-3}C_{n-1}^{2n-3}$	228	1001	4784

Throughout this paper, we follow the standard terminologies and notations used in the representation theory of algebras, see [3, 4, 16].

3 Complements of partial tilting modules

Let A be a finite dimensional algebra over an algebraically closed field k . In this section, we prove Theorem A and give a new proof of that every almost complete tilting module has at most two complements.

Let M be a partial tilting A -module. It has been proved in [5] that M has a complement Y , which is called the Bongartz complement. This complement is constructed by a universal sequence $0 \rightarrow A \rightarrow E \rightarrow M^s \rightarrow 0$, where $s = \dim_k \text{Ext}_A^1(M, A)$ and $E = Y^t \oplus M'$ with $M' \in \text{add } M$ and some integer t .

Note that M is also a τ -rigid A -module. By [1, Theorem 2.10], there exists a τ -rigid A -module X such that $T = M \oplus X$ is τ -tilting and $\text{Fac } T = {}^\perp(\tau M)$. X is called the Bongartz τ -tilting complement of M and it is unique up to isomorphism. We prove that X coincides with the Bongartz complement Y .

Theorem 3.1. *Let M be a partial tilting A -module and X be its Bongartz τ -tilting complement. Then $\text{pd}_A X \leq 1$ and $T = M \oplus X$ is a tilting A -module. In particular, X coincides with the Bongartz complement of M .*

Proof. Note that $\text{pd}_A M \leq 1$ since M is a partial tilting A -module. Then we have $\text{Hom}_A(DA, \tau M) = 0$. This implies that $DA \in {}^\perp(\tau M) = \text{Fac } T$ and T is faithful. By Lemma 2.1, T is a tilting A -module and $\text{pd}_A X \leq 1$.

We claim that X is the Bongartz complement of M . In fact, assume $X = \bigoplus_{i=1}^r X_i$ is basic and $T[i] = M \oplus X[i]$. By [15, Proposition 1.2], we only need to show that there is no surjection from any module in $\text{add } T[i]$ to X_i for $i = 1, 2, \dots, r$. If there exists such a surjection, X_i is generated by $T[i]$ and $\text{Fac } T = \text{Fac } T[i] = {}^\perp(\tau M)$. This implies that $X[i]$ is also the Bongartz τ -tilting complement of M , a contradiction. \square

Remark. By Lemma 2.6, we have a short exact sequence $0 \rightarrow A \xrightarrow{f} T_1 \xrightarrow{g} T_2 \rightarrow 0$ with $T_1, T_2 \in \text{add } T$ and $\text{add } T_1 \cap \text{add } T_2 = 0$. f is injective since A is cogenerated by T . Let us show that $X \in \text{add } T_1$. It is obvious that all $X_i \in \text{add}(T_1 \oplus T_2)$ since T is a tilting A -module. If there exists some $X_i \in \text{add } T_2$, then X_i is generated by T_1 and then by $T[i]$ since $\text{add } T_1 \cap \text{add } T_2 = 0$. This contradicts the fact that X is the Bongartz complement of M . As a result, $X \in \text{add } T_1$ and $T_2 \in \text{add } M$. This short exact sequence is the universal sequence constructed in [5].

Let M be an almost complete tilting A -module. Then M has at most two complements and it has exactly two complements if and only if it is faithful (see [15, 6]). By using the mutation of support τ -tilting modules, we give a new proof of these results.

Theorem 3.2. [6, Proposition 2.3] *Let M be an almost complete tilting A -module. Then M has exactly two complements if it is faithful. Otherwise, it has only one complement.*

Proof. Let X be the Bongartz complement of M . $(M, 0)$ is an almost complete support τ -tilting pair. By Lemma 2.3, it is a direct summand of exactly two support τ -tilting pairs. Obviously, one is $(M \oplus X, 0)$ and the other is of the form $(M \oplus Y, 0)$ with Y indecomposable and $M \oplus Y$ τ -tilting or (M, P) with P projective and $\text{Hom}_A(P, M) = 0$. In the first case, by Lemma 2.4, there exists an exact sequence $X \rightarrow M' \rightarrow Y^s \rightarrow 0$ with $M' \in \text{add} M$ and some integer s . Note that if a tilting A -module T contains M as a direct summand, then the support τ -tilting pair $(T, 0)$ contains $(M, 0)$ as a direct summand. Thus M has at most two complements.

(a) Assume M is faithful. Then M is sincere and $\text{Hom}_A(P, M) \neq 0$ for all projective A -modules P . So the other support τ -tilting pair is $(M \oplus Y, 0)$ and $M \oplus Y$ is a tilting A -module since it is faithful. Thus M has exactly two complements X and Y .

(b) Assume M is not faithful. If M is not sincere, then $M \oplus Y$ is not sincere since Y is generated by M . This implies that $M \oplus Y$ is not τ -tilting because all τ -tilting modules are sincere. Consequently the other support τ -tilting pair is (M, P) and M has only one complement.

If M is sincere, the other support τ -tilting pair is $(M \oplus Y, 0)$. We claim that $M \oplus Y$ is not tilting. Otherwise, A is cogenerated by $M \oplus Y$. Let $g : A \rightarrow F$ be an injection with $F \in \text{add}(M \oplus Y)$. Since Y is generated by M , there exists a surjection $h : E \rightarrow F$ with $E \in \text{add} M$. Since A is projective there exists $f : A \rightarrow E$ with $g = hf$, hence f is injective and A is cogenerated by M , which contradicts the assumption that M is not faithful. In this case M has only one complement. \square

Let X and Y be two nonisomorphic complements of an almost complete tilting A -module M . It is shown in [6] that they are connected by a nonsplit short exact sequence $0 \rightarrow X \xrightarrow{f} M' \xrightarrow{g} Y \rightarrow 0$. Now we give a different way to construct this sequence.

Theorem 3.3. [6, Theorem 1.1] *Let X and Y be two nonisomorphic complements of an almost complete tilting A -module M and $\text{Ext}_A^1(Y, X) \neq 0$. Then there exists a nonsplit short exact sequence $0 \rightarrow X \xrightarrow{f} M' \xrightarrow{g} Y \rightarrow 0$, where f is a minimal left $\text{add} M$ -approximation and g is a minimal right $\text{add} M$ -approximation.*

Proof. Let X be the Bongartz complement of M . From the proof of Theorem 3.2, we know there exists an exact sequence $X \xrightarrow{f} M' \xrightarrow{g} Y^s \rightarrow 0$ with $M' \in \text{add} M$ and some integer s . Moreover, f is a minimal left $\text{add} M$ -approximation of X and g is a right $\text{add} M$ -approximation of Y^s .

Firstly, we prove f is an injection. This only needs to show X is cogenerated by M . By the remark after Theorem 3.1, we get a short exact sequence $0 \rightarrow A \rightarrow (M \oplus X)' \rightarrow M'' \rightarrow 0$ with $(M \oplus X)' \in \text{add}(M \oplus X)$ and $M'' \in \text{add}M$. Note that M is faithful since it has two nonisomorphic complements. Let $\varphi : A \rightarrow F$ be an injection with $F \in \text{add}M$. Then we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & (M \oplus X)' & \longrightarrow & M'' \longrightarrow 0 \\ & & \downarrow \varphi & & \downarrow h & & \parallel \\ 0 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & M'' \longrightarrow 0 \end{array}$$

The lower sequence splits since M has no self-extension, thus $E \cong F \oplus M''$. Note that φ is injective, by snake lemma, h is an injection. Consequently $(M \oplus X)'$ is cogenerated by M and then X is cogenerated by M .

Secondly, we show g is right minimal, that is every $t \in \text{End} M'$ such that $gt = g$ is an automorphism. Then there exists an endomorphism μ of X that makes the following diagram commute. If μ is not an isomorphism, it must be nilpotent since X is indecomposable and $\text{End} X$ is local. So there

$$\begin{array}{ccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & M' & \xrightarrow{g} & Y^s \longrightarrow 0 \\ & & \downarrow \mu & & \downarrow t & & \parallel \\ 0 & \longrightarrow & X & \xrightarrow{f} & M' & \xrightarrow{g} & Y^s \longrightarrow 0 \end{array}$$

exists some integer m such that $\mu^m = 0$. Then $t^m f = f\mu^m = 0$ and so t^m factors through Y^s , that is, there exists $\alpha : Y^s \rightarrow M'$ such that $t^m = \alpha g$. Because $gt^m = g$, we deduce that $g\alpha g = g$ and consequently $g\alpha = 1_{Y^s}$ since g is a surjection. This contradicts the fact that the sequence is not split. Thus μ is an isomorphism and so is t .

Finally, we claim that $s = 1$. Let $h : M_0 \rightarrow Y$ be a minimal right add M -approximation of Y and $N = \text{Ker } h$. Then the map

$$\psi = \begin{pmatrix} h & 0 \\ & \ddots \\ 0 & h \end{pmatrix} : M_0^s \longrightarrow Y^s$$

is a right add M -approximation of Y^s . Thus there is a decomposition $M_0^s = M' \oplus M_1$ such that $\psi = (g, 0)^t$. So there exists a map $\theta : N^s \rightarrow X \oplus M_1$ that makes the following

diagram commute.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N^s & \longrightarrow & M_0^s & \xrightarrow{\psi} & Y^s & \longrightarrow & 0 \\
& & \downarrow \theta & & \parallel & & \parallel & & \\
0 & \longrightarrow & X \oplus M_1 & \xrightarrow{\phi} & M' \oplus M_1 & \xrightarrow{\psi} & Y^s & \longrightarrow & 0
\end{array}$$

where

$$\phi = \begin{pmatrix} f & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that θ is an isomorphism and $N^s \cong X \oplus M_1$. Thus we get $s = 1$ since $X \notin \text{add } M_1$. \square

4 Tilting quiver and support τ -tilting quiver

Let A be a finite dimensional algebra over an algebraically closed field k . In this section, we give a new proof of that the Hasse quiver associated to the poset of basic tilting A -modules coincides with the tilting quiver $\vec{Q}(\text{tilt } A)$. Moreover, when A is hereditary, we calculate the number of arrows in $\vec{Q}(\text{tilt } A)$ and prove Theorem B.

Riedtmann and Schofield define the tilting quiver $\vec{Q}(\text{tilt } A)$ in [15] as follows. The vertices are the isomorphism classes of basic tilting modules. There is an arrow $T_1 \rightarrow T_2$ if $T_1 = M \oplus X$, $T_2 = M \oplus Y$ with X, Y indecomposable and there exists a short exact sequence $0 \rightarrow X \rightarrow M' \rightarrow Y \rightarrow 0$ with $M' \in \text{add } M$. On the other hand, the set of basic tilting modules has a natural partial order given by $T_1 > T_2$ if $\text{Fac } T_1 \supseteq \text{Fac } T_2$. Happel and Unger have proved in [8] that the Hasse quiver associated to the poset of basic tilting modules coincides with the tilting quiver $\vec{Q}(\text{tilt } A)$.

Note that tilting A -modules are also the vertices in the support τ -tilting quiver $\vec{Q}(\text{stilt } A)$. Then we prove Happel and Unger's result in [8] from the viewpoint of support τ -tilting modules.

Theorem 4.1. [8, Theorem 4.1] *The tilting quiver $\vec{Q}(\text{tilt } A)$ is the Hasse quiver of the poset of tilting A -modules.*

Proof. Let $T_1 \rightarrow T_2$ be an arrow in $\vec{Q}(\text{tilt } A)$. Then we assume that $T_1 = M \oplus$

$X, T_2 = M \oplus Y$ with X, Y indecomposable and there exists a short exact sequence $0 \rightarrow X \rightarrow M' \rightarrow Y \rightarrow 0$ with $M' \in \text{add } M$. It is obvious that $\text{Fac } T_2 = \text{Fac } (M \oplus Y) \subseteq \text{Fac } M \subseteq \text{Fac } (M \oplus X) = \text{Fac } T_1$. Now we show the inclusion is minimal. If there exists a tilting A -module T_3 such that $\text{Fac } T_2 \subseteq \text{Fac } T_3 \subseteq \text{Fac } T_1$, then by [1, Proposition 2.26], we have $\text{add } T_1 \cap \text{add } T_2 \subseteq \text{add } T_3$. Since $\text{add } T_1 \cap \text{add } T_2 = \text{add } M$, we know $T_3 = M \oplus X$ or $T_3 = M \oplus Y$.

Let $\text{Fac } T_2 \subseteq \text{Fac } T_1$ be a minimal inclusion, that is there is no tilting A -module T_3 ($T_3 \not\cong T_1, T_2$) such that $\text{Fac } T_2 \subseteq \text{Fac } T_3 \subseteq \text{Fac } T_1$. Note that $T_1, T_2 \in \vec{Q}(\text{stilt } A)_0$. Assume there exists a support τ -tilting A -module T such that $\text{Fac } T_2 \subseteq \text{Fac } T \subseteq \text{Fac } T_1$. If $a \in A$ satisfies $a \text{Fac } T = 0$, then we have $a \text{Fac } T_2 = 0$. According to [1, Corollary 2.8], there is a bijection $T \rightarrow \text{Fac } T$ between basic tilting modules and faithful functorially finite torsion classes. Then we get $a = 0$, and this implies that T is a tilting A -module, a contradiction. Thus the inclusion $\text{Fac } T_2 \subseteq \text{Fac } T_1$ is minimal with respect to the partial order of support τ -tilting A -modules. As support τ -tilting A -modules, T_2 is a left mutation of T_1 since $\text{Fac } T_2 \subseteq \text{Fac } T_1$. By Lemma 2.4 and Theorem 3.3, there exists a short exact sequence $0 \rightarrow X \rightarrow M' \rightarrow Y \rightarrow 0$ with $M' \in \text{add } M$ and $T_1 = M \oplus X, T_2 = M \oplus Y$. It follows that there is an arrow $T_1 \rightarrow T_2$ in $\vec{Q}(\text{tilt } A)$. \square

From the proof of Theorem 4.1 we can get the following result.

Theorem 4.2. *The tilting quiver $\vec{Q}(\text{tilt } A)$ can be embedded into the support τ -tilting quiver $\vec{Q}(\text{stilt } A)$.*

From now on, we assume that $A = kQ$ is a finite dimensional hereditary algebra. In general, the tilting quiver $\vec{Q}(\text{tilt } A)$ of A may not be connected. For example, the tilting quiver $\vec{Q}(\text{tilt } A)$ is two disjoint rays when A is the Kronecker algebra. However, the support τ -tilting quiver $\vec{Q}(\text{stilt } A)$ of A is always connected.

Theorem 4.3. *Let A be a finite dimensional hereditary algebra. Then the support τ -tilting quiver $\vec{Q}(\text{stilt } A)$ of A is connected.*

Proof. Let \bar{A} be the duplicated algebra of a hereditary algebra A and \bar{P} be the direct sum of all nonisomorphic indecomposable projective-injective \bar{A} -modules. For an \bar{A} -module M , we denote by $\Omega_{\bar{A}}^{-1}M$ and $\Omega_{\bar{A}}^{-1}M$ respectively its first syzygy and first cosyzygy. We set $\Sigma_1 = \{ \Omega_{\bar{A}}^{-1}P \mid P \text{ is an indecomposable projective } A\text{-module} \}$. Let T be a tilting \bar{A} -module, we have a decomposition $T = T_1 \oplus T_2 \oplus \bar{P}$ with $T_1 \in \text{mod-}A$ and $T_2 \in \text{add } \Sigma_1$. By [2, Theorem 10], we have a bijection between tilting \bar{A} -modules and cluster tilting

objects in \mathcal{C}_A . On the other hand, by Lemma 2.5, we get a bijection between cluster tilting objects in \mathcal{C}_A and support τ -tilting A -modules since A is a cluster tilting object in \mathcal{C}_A . Thus there is a bijection between tilting \overline{A} -modules and support τ -tilting A -modules, sending $T = T_1 \oplus T_2 \oplus \overline{P}$ to $(T_1, \Omega_{\overline{A}} T_2)$.

Then we prove there is a quiver isomorphism between $\vec{Q}(\text{tilt } \overline{A})$ and $\vec{Q}(\text{s}\tau\text{-tilt } A)$. It only needs to show the Hasse quiver of the poset of tilting \overline{A} -modules corresponds to that of support τ -tilting A -modules.

Let T and T' be tilting \overline{A} -modules and $\text{Fac } T' \subseteq \text{Fac } T$. Then we have $T'_1 \in \text{Fac } (T_1 \oplus T_2 \oplus \overline{P})$. Since $T_1, T'_1 \in \text{mod-}A$ and $T_2, \overline{P} \notin \text{mod-}A$, we get $T'_1 \in \text{Fac } T_1$ and then $\text{Fac } T'_1 \subseteq \text{Fac } T_1$.

Conversely, assume $\text{Fac } T'_1 \subseteq \text{Fac } T_1$. Since $T'_2 \notin \text{mod-}A$, we have $T'_2 \in \text{Fac } \overline{P}$. This implies that $T'_1 \oplus T'_2 \oplus \overline{P} \in \text{Fac } (T_1 \oplus T_2 \oplus \overline{P})$ and then $\text{Fac } T' \subseteq \text{Fac } T$.

According to [19, Proposition 4.1], we know that the tilting quiver $\vec{Q}(\text{tilt } \overline{A})$ of \overline{A} is connected, and hence the support τ -tilting quiver $\vec{Q}(\text{s}\tau\text{-tilt } A)$ is connected. \square

Example. Let $A = kQ$ be the Kronecker algebra with $Q : 1 \rightrightarrows 2$. Then the support $\vec{Q}(\text{s}\tau\text{-tilt } A)$ is as follows.

$$\dots \rightarrow \begin{smallmatrix} 222 \\ 11 \end{smallmatrix} \oplus \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \rightarrow \begin{smallmatrix} 22 \\ 1 \end{smallmatrix} \oplus 2 \rightarrow 2 \rightarrow 0 \leftarrow 1 \leftarrow 1 \oplus \begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \rightarrow \begin{smallmatrix} 2 \\ 11 \end{smallmatrix} \oplus \begin{smallmatrix} 22 \\ 111 \end{smallmatrix} \rightarrow \dots$$

Let Q be a Dynkin quiver with n vertices and $A = kQ$ be the path algebra. It is known that the number $a_n(Q)$ of basic tilting A -modules is independent of the orientation of Q . This implies that the number of vertices in $\vec{Q}(\text{tilt } A)$ is a constant for all Dynkin quivers of the same type. By [12, Theorem 0.1], the number of arrows in $\vec{Q}(\text{tilt } A)$ is also a constant. By using the support τ -tilting quiver $\vec{Q}(\text{s}\tau\text{-tilt } A)$, we give a new method to calculate the number of these arrows.

Corollary 4.4. [12, Theorem 0.1] Let Q be a Dynkin quiver with n vertices and $A = kQ$. Then the number of arrows in $\vec{Q}(\text{tilt } A)$ (denoted by $\#\vec{Q}(\text{tilt } A)_1$) does not depend on the

orientation of Q . In particular,

$$\#\vec{Q}(\text{tilt } A)_1 = \begin{cases} C_{2n-1}^{n+1} & \text{if } Q = A_n \\ (3n-4)C_{2n-4}^{n-3} & \text{if } Q = D_n \\ 1140 & \text{if } Q = E_6 \\ 8008 & \text{if } Q = E_7 \\ 66976 & \text{if } Q = E_8 \end{cases}$$

Proof. We regard $\vec{Q}(\text{tilt } A)$ as a subquiver of $\vec{Q}(s\tau\text{-tilt } A)$. By Lemma 2.3, each vertex in $\vec{Q}(s\tau\text{-tilt } A)$ has exactly n neighbours. Let T be a tilting A -module, then the neighbours of T in $\vec{Q}(s\tau\text{-tilt } A)$ are tilting A -modules or support τ -tilting A -modules with $n-1$ nonisomorphic indecomposable direct summands. Note that each support τ -tilting A -modules with $n-1$ nonisomorphic indecomposable direct summands is connected with exactly one tilting A -module by an arrow in $\vec{Q}(s\tau\text{-tilt } A)$. Then we get that

$$\#\vec{Q}(\text{tilt } A)_1 = \frac{1}{2}(a_n(Q) \times n - a_{n-1}(Q)).$$

By Lemma 2.6, we can calculate the number of arrows in $\vec{Q}(\text{tilt } A)$ and this number is independent of the orientation of Q . \square

5 Non-saturated vertices in tilting quiver

Let Q be a quiver with n vertices and $A = kQ$ be the finite dimensional hereditary algebra over an algebraically closed field k . In this section, by using support τ -tilting quiver, we give new proofs for some Happel and Unger's results. Moreover, we prove the conjecture of Happel and Unger in [9] when Q is a Dynkin quiver, a Euclidean quiver and a wild quiver with two or three vertices, and we also provide a counterexample for this conjecture when Q is a wild quiver with four vertices.

Let T be a tilting A -module, we denote by $s(T)$ (respectively $e(T)$) the number of arrows starting (respectively ending) at T in the tilting quiver $\vec{Q}(\text{tilt } A)$. For a support τ -tilting A -module M , by Lemma 2.3, the number of arrows starting or ending at M in $\vec{Q}(s\tau\text{-tilt } A)$ is equal to n . Since $\vec{Q}(\text{tilt } A)$ can be embedded into $\vec{Q}(s\tau\text{-tilt } A)$, we have $s(T) + e(T) \leq n$. We say T is saturated if $s(T) + e(T) = n$.

The following result in [9] is a sufficient and necessary condition for a tilting A -module

to be saturated in $\vec{Q}(\text{tilt}A)$. Here we give a new proof by using support τ -tilting quiver.

Theorem 5.1. [9, Propostion 3.2] *Let T be a basic tilting A -module. It is saturated if and only if $(\underline{\dim} T)_i \geq 2$ for all $1 \leq i \leq n$.*

Proof. Assume that $T = \bigoplus_{i=1}^n T_i$ is saturated and there is some i with $(\underline{\dim} T)_i = 1$. Then there must be an indecomposable summand T_k of T such that $(\underline{\dim} T_k)_i = 1$. So we have a decomposition $T = T[k] \oplus T_k$ with $(\underline{\dim} T[k])_i = 0$. This implies that $T[k]$ is a non-sincere almost complete tilting A -module and it has only one complement. Then T is not saturated, a contradiction.

Conversely, assume $(\underline{\dim} T)_i \geq 2$ for all $1 \leq i \leq n$. If T is not saturated, there exists an arrow $T \rightarrow (M, P)$ in $\vec{Q}(\text{s}\tau\text{-tilt}A)$ where $T = M \oplus X$ with X indecomposable and P is an indecomposable projective A -module. By Lemma 2.6, we get a short exact sequence $0 \rightarrow P \xrightarrow{f} T_1 \xrightarrow{g} T_2 \rightarrow 0$ with $T_1, T_2 \in \text{add} T$ and $\text{add} T_1 \cap \text{add} T_2 = 0$. f is an injection since P is cogenerated by T . Note that $f \neq 0$ and $\text{Hom}_A(P, M) = 0$, then we get $T_1 = X^s \oplus M_1$ for some integer s and $M_1, T_2 \in \text{add} M$. Applying $\text{Hom}_A(P, -)$ to the above short exact sequence, we get $\text{Hom}_A(P, T_1) \cong \text{Hom}_A(P, P) \cong k$. This implies that $s = 1$ and $(\underline{\dim} X)_i = 1$ for some integer $i \in (1, n)$. It is obvious that $(\underline{\dim} M)_i = 0$, then we have $(\underline{\dim} T)_i = (\underline{\dim} M)_i + (\underline{\dim} X)_i = 1$, a contradiction. \square

Remark. *Let i be a source vertex of Q_0 and $A = \bigoplus_{i=1}^n P_i$. Then we have $(\underline{\dim} \bigoplus_{j \neq i} P_j)_i = 1$. By the above theorem, we know A is not saturated. Dually, DA is not saturated either.*

Recall that the tilting quiver $\vec{Q}(\text{tilt}A)$ can be regarded as a subquiver of $\vec{Q}(\text{s}\tau\text{-tilt}A)$, then we prove the following result which is contained in [17].

Theorem 5.2. [17, Theorem 3.1] *Each connected component of the tilting quiver $\vec{Q}(\text{tilt}A)$ contains a non-saturated vertex.*

Proof. If $\vec{Q}(\text{tilt}A)$ is connected, then A is one of the non-saturated vertices in $\vec{Q}(\text{tilt}A)$. Now assume $\vec{Q}(\text{tilt}A)$ is not connected. If $\vec{Q}(\text{tilt}A)$ has a connected component \mathcal{R} which contains only saturated vertices, choose a vertex T in \mathcal{R} . Since $\vec{Q}(\text{tilt}A)$ can be embedded into $\vec{Q}(\text{s}\tau\text{-tilt}A)$ which is connected, there is a path $A = T_n - \cdots - T_2 - T_1 - T_0 = T$ in the underlying graph $Q(\text{s}\tau\text{-tilt}A)$ where T_i are support τ -tilting A -modules for all $0 \leq i \leq n$. Since A is not contained in \mathcal{R} , there must exist support τ -tilting A -modules in this path which are not tilting. Choose a minimal vertex T_j in this path such that T_j is a proper support τ -tilting A -module and T_i is tilting for all $0 \leq i \leq j-1$. Note that T_{j-1} is

saturated since it is in \mathcal{R} , and this implies that the number of arrows starting or ending at T_{j-1} in $\vec{Q}(\text{stilt}A)$ is more than n , a contradiction. This completes the proof. \square

D.Happel and L.Unger conjecture in [9] that each connected component of $\vec{Q}(\text{tilt}A)$ contains only finitely many non-saturated vertices. Firstly we prove that this conjecture is true if Q is a Dynkin or Euclidean quiver.

Theorem 5.3. *Let $A = kQ$ be a finite dimensional hereditary algebra. If Q is Dynkin or Euclidean type, then each connected component of $\vec{Q}(\text{tilt}A)$ contains finitely many non-saturated vertices.*

Proof. Let $A = kQ$ be a finite dimensional hereditary algebra. If Q is a Dynkin quiver, then A is a representation-finite algebra. So $\vec{Q}(\text{tilt}A)$ is finite and our result is true.

Assume Q is a Euclidean quiver. If a tilting A -module T is not saturated, there must be an arrow $T \rightarrow (M, P)$ in $\vec{Q}(\text{stilt}A)$ where $T = M \oplus X$ with X indecomposable and P is an indecomposable projective A -module. Then M is a tilting kQ_i -module where Q_i is a quiver obtained by removing a vertex i from Q and all arrows connected with i . Thus each non-saturated tilting A -module contains a tilting kQ_i -module as a direct summand. Since all path algebras kQ_i for $1 \leq i \leq n$ are representation-finite, there are only finitely many tilting kQ_i -modules. This implies that there are only finitely many non-saturated tilting A -modules. Then we get our result when Q is a Euclidean quiver. \square

Before we prove this conjecture for Q being a wild quiver with two or three vertices, we introduce the following lemma in [18].

Lemma 5.4. [18, Main Theorem] *Let $A = kQ$ be a finite dimensional hereditary algebra where Q is a wild quiver with three vertices and e be a primitive idempotent in A . Let regular A -module M be a tilting $A/\langle e \rangle$ -module and $M \oplus X$ be a tilting A -module. If $T \not\cong M \oplus X$ is a vertex in the connected component of $\vec{Q}(\text{tilt}A)$ containing $M \oplus X$, then T has at least two sincere indecomposable direct summands and each sincere indecomposable direct summand of T is τ -sincere.*

Theorem 5.5. *Let $\Gamma = kQ$ be a finite dimensional hereditary algebra. If Q is a wild quiver with two or three vertices, then each connected component of $\vec{Q}(\text{tilt}\Gamma)$ contains finitely many non-saturated vertices.*

Proof. If Q is a wild quiver with two vertices, it is of the form $2 \begin{smallmatrix} \rightrightarrows \\ \rightleftarrows \end{smallmatrix} 1$ with at least three arrows. By [14, XVIII, Corollary 2.16], there are no regular tilting Γ -modules and

all tilting Γ -modules are preprojective or preinjective. The tilting quiver $\vec{Q}(\text{tilt}\Gamma)$ is of the form

$$\circ \rightarrow \circ \rightarrow \circ \rightarrow \dots$$

$$\circ \leftarrow \circ \leftarrow \circ \leftarrow \dots$$

It is easy to see that each connected components of $\vec{Q}(\text{tilt}\Gamma)$ contains exactly one non-saturated vertex.

Assume Q is a wild quiver with three vertices and $T = T_1 \oplus T_2 \oplus T_3$ is a basic tilting Γ -module. If T is a non-saturated vertex in $\vec{Q}(\text{tilt}\Gamma)$, then there exists an arrow $T \rightarrow (T_1 \oplus T_2, P)$ in $\vec{Q}(\text{st-tilt}\Gamma)$ where P is an indecomposable projective Γ -module. Let e be the primitive idempotent in Γ corresponding to P . Then $T_1 \oplus T_2$ is a tilting $\Gamma/\langle e \rangle$ -module and each non-saturated tilting Γ -module contains a tilting $\Gamma/\langle e \rangle$ -module as a direct summand.

If $\Gamma/\langle e \rangle$ is a representation-finite algebra, we can find only finitely many non-saturated tilting Γ -modules which contain tilting $\Gamma/\langle e \rangle$ -modules as direct summands.

If $\Gamma/\langle e \rangle$ is a representation-infinite algebra, the quiver of $\Gamma/\langle e \rangle$ is of the form $\circ \rightleftarrows \circ$ with at least two arrows. Since there are only finitely many non-sincere indecomposable preprojective and preinjective Γ -modules, all but finitely many tilting $\Gamma/\langle e \rangle$ -modules are regular Γ -modules. Thus all but finitely many non-saturated tilting Γ -modules contain tilting $\Gamma/\langle e \rangle$ -modules which are regular Γ -modules as direct summands. Assume $T_1 \oplus T_2$ is a regular Γ -module. By Lemma 5.4, T is contained in a connected component of $\vec{Q}(\text{tilt}\Gamma)$ which has only one non-saturated vertex T . This completes our proof. \square

Remark. We should mention that the conjecture of Happel and Unger is not true for some wild quivers.

In order to provide a counterexample, we need the following lemma.

Lemma 5.6. [17, Theorem 1] *Let M be a partial tilting A -module with $n-2$ nonisomorphic indecomposable direct summands and $\vec{Q}(\text{tilt}_M A)$ be the subquiver of $\vec{Q}(\text{tilt} A)$ with vertices T such that M is a direct summand of T . If M is not sincere and $\vec{Q}(\text{tilt}_M A)$ is infinite, then $\vec{Q}(\text{tilt}_M A)$ is of the form*

$$\circ \rightarrow \circ \rightarrow \circ \rightarrow \dots$$

$$\circ \leftarrow \circ \leftarrow \circ \leftarrow \dots$$

The following example is taken from [17] which is a counterexample to the conjecture of Happel and Unger.

Example. Let $B = kQ$ be the path algebra of the wild quiver $Q : 1 \rightleftharpoons 2 \leftarrow 3 \rightarrow 4$. We claim that the tilting quiver $\vec{Q}(\text{tilt} B)$ contains a connected component which has infinitely many non-saturated vertices.

Indeed, we assume that N is a tilting module over the Kronecker algebra $k(1 \rightleftharpoons 2)$ and it has no nonzero projective direct summands. Let $I_3 = 3$ and $I_4 = \begin{smallmatrix} 3 \\ 4 \end{smallmatrix}$. Then $I_3 \oplus I_4 \oplus \tau_B N$ is a tilting B -module. The Coxeter matrix of B is

$$\Phi_B = \begin{pmatrix} -1 & 2 & 0 & 0 \\ -2 & 3 & 1 & 0 \\ -2 & 3 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

By $\underline{\dim} \tau_B N = \Phi_B \underline{\dim} N$, we know $(\underline{\dim} \tau_B N)_4 = 0$. Thus we get that $(\underline{\dim} I_4 \oplus I_3 \oplus \tau_B N)_4 = 1$ and $I_4 \oplus I_3 \oplus \tau_B N$ is not saturated. Since there are infinitely many tilting modules over the Kronecker algebra, by Lemma 5.6, at least one of the connected components in $\vec{Q}(\text{tilt}_{I_3 \oplus I_4} B)$ contains infinitely many non-saturated vertices and we denote this component by \mathcal{G} . Then the connected component in $\vec{Q}(\text{tilt} B)$ which contains \mathcal{G} has infinitely many non-saturated vertices.

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